§6.1: Absolute Extrema

Absolute Minimum and Maximum

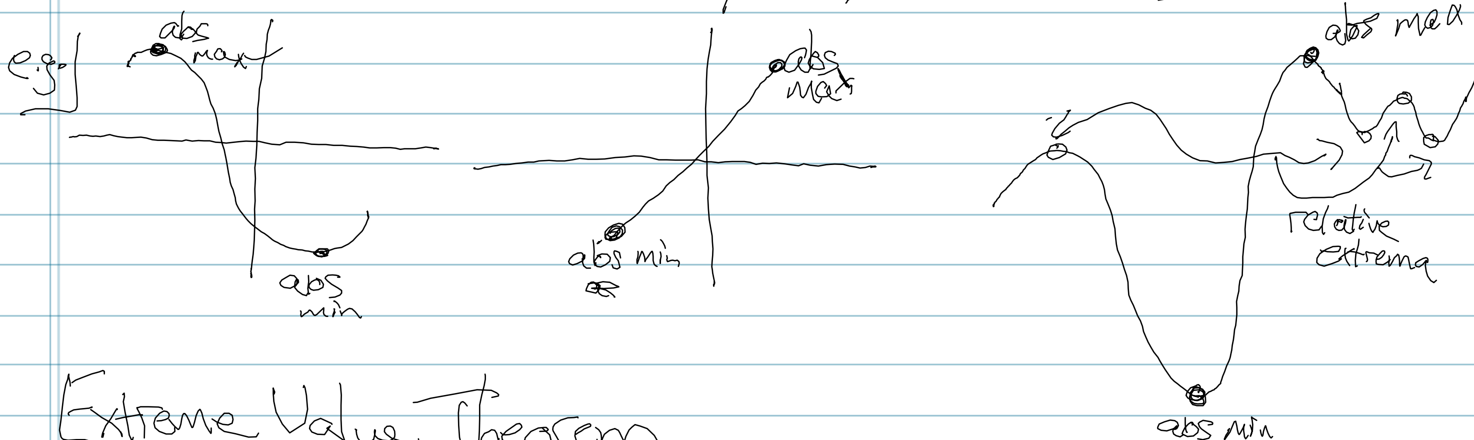
Let f be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is:

(1) the absolute maximum of f on the interval if $f(c) \geq f(x)$, $\forall x$ on the interval

(2) the absolute minimum of f on the interval if ~~f~~ $f(c) \leq f(x)$, $\forall x$

- We may refer to these collectively as absolute extremum

e.g.,



Extreme Value Theorem

A function, f, that is continuous on a closed interval $[a,b]$ will have both an absolute max and an absolute min, on the interval.

Finding Absolute Extrema

To find absolute extrema for a function, f, continuous on a closed interval $[a,b]$

(1.) Find all critical #'s for f in (a,b)

(2.) Evaluate f for all critical numbers in (a,b)

(3.) Evaluate f at the endpoints a,b of $[a,b]$ (i.e. find $f(a)$, $f(b)$)

(4). The largest value found in step 2 or 3 is the absolute max and the smallest is the absolute min for f on $[a,b]$

e.g. find the absolute extrema of the function

(A) $f(x) = 5x^{2/3} + 2x^{5/3}$ on $[-2,1]$

First, find the derivative and get critical points:

$f'(x) = \left(\frac{2}{3}\cdot 5\right)x^{-1/3} + \left(\frac{5}{3}\cdot 2\right)x^{2/3} = \frac{10}{3}x^{-1/3}\left[1+x\right] = \frac{10(1+x)}{3(x^{1/3})}$

So $f'(x)$ is 0 or undefined when $x_c = \{-1, 0\}$

Now we must evaluate at each x_c and our endpoints:

$f(-2) = 5(-2)^{2/3} + 2(-2)^{5/3} \approx 1.587$

$f(0) = 0$ min

$f(1) = 5+2 = 7$ max $\Rightarrow$ abs. max @ $(1,7)$, abs min @ $(0,0)$

$f(-1) = 5-2 = 3$

(B) $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ on $(-\infty, \infty)$

Take the derivative, find critical points

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2-x-2) = 12x(x+1)(x-2)$

$\Rightarrow x_c = \{0, -1, 2\}$ and we also must check endpoints:

$f(0) = 2$

$f(2) = -30$ min

$f(-1) = -3$

endpts (must use limits since unbounded interval)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 3x^4 - 4x^3 - 12x^2 + 2 = \lim_{x\to\infty} 3x^4 = \infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3x^4 - 4x^3 - 12x^2 + 2 = \lim_{x\to-\infty} 3x^4 = \infty$

So NO absolute max!

abs min @ $(2, -30)$

Critical Point Theorem

Suppose a function f is continuous on an interval I, and that f has exactly one critical number on the interval I, located at $x=c$

- If f has a relative max at $x=c$, then this relative max is the absolute max of f on the interval
- If f has a relative minimum at $x=c$, then this relative minimum is the absolute minimum of f on the interval I.

§6.2 : Applications of Extrema

Strategy for Solving Applied Extrema Problem

- sketches are often helpful (i.e. diagrams)
- immediately find the variable to be minimized or maximized
- beware of domains (don't look for extrema in non-realistic ranges)
- we look for critical points (numbers) in the maximized variable as a function of some single, other variable.
- If the domain has closed endpoints, evaluate them and the critical points to find max/min
- If the domain has open endpoints, apply the critical point theorem, if only one critical #
  ↳ otherwise, evaluate all critical #'s and evaluate endpoint limits

e.g.) (a) Find two non-negative #'s, x and y, for which $2x+y=30$ and xy^2 is maximized.

What's maximized? Set $xy^2 = M$

Constraint? $2x+y=30 \Rightarrow x = 15 - \frac{y}{2}$

Sub this value into M in order to get M as a function of 1 variable

$M(y) = \left(15 - \frac{y}{2}\right)y^2 = 15y^2 - \frac{y^3}{2}$; Now find critical #'s

$M'(y) = 30y - \frac{3}{2}y^2 = 0 \Rightarrow 3y\left(10 - \frac{y}{2}\right) \Rightarrow 3y = 0$ or $10 - \frac{y}{2} = 0$

$\Rightarrow y_c = \{0, 20\}$

What is Domain? Must have $x \geq 0$, so

$x = 15 - \frac{y}{2} \geq 0 \Rightarrow \frac{y}{2} \leq 15 \Rightarrow y \leq 30$,

This means $y=30$ and $y=0$ are endpoints of our domain:

Now we check all endpoints and critical values:

$M(0) = 0$

$M(20) = 15(20)^2 - \frac{20^3}{2} = 6000 - \frac{8000}{2} = 2000$ ← max

$M(30) = 15(30)^2 - \frac{30^3}{2} = 13500 - 13500 = 0$

So $y=20$ maximizes $M = xy^2$.

What is x?, use our constraint:

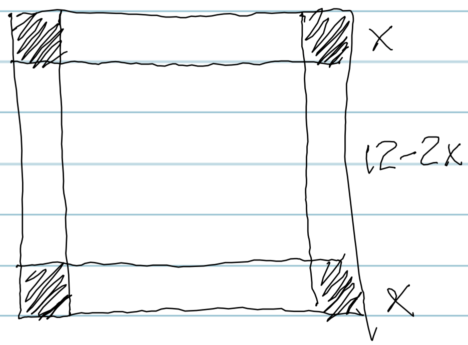
$x = 15 - \frac{y}{2} = 15 - \frac{20}{2} = 5$

$\Rightarrow (5, 20)$ maximizes $M = xy^2$ given $x = 15 - \frac{y}{2}$ (ie $2x+y=30$)

(B) An open box is to be made by cutting a square corner from a 12in by 12in piece of metal, and then folding up the sides.

- what sidelength (square) should be cut from the corners to maximize volume?

picture:



What do we maximize? Volume of box

$V = L \cdot W \cdot H$

Here $H = x$; $L = W = 12-2x$

$\Rightarrow V = x(12-2x)^2 = x(2(6-x))^2$

$= 4x(x^2-12x+36)$

$= 4(x^3-12x^2+36x)$

- Now we must maximize volume, $V(x)$, so take a derivative.

$V'(x) = 4[3x^2-24x+36] = 12(x^2-8x+12) = 12(x-6)(x-2)$

$\Rightarrow x_c = \{2, 6\}$

can't have negative width!

- Domain? $x \geq 0$, but also $2x \leq 12 \Rightarrow x \leq 6$

Now we must check the points $\{0, 2, 6\}$

$V(x) = x(12-2x)^2 = 4x(6-x)^2$

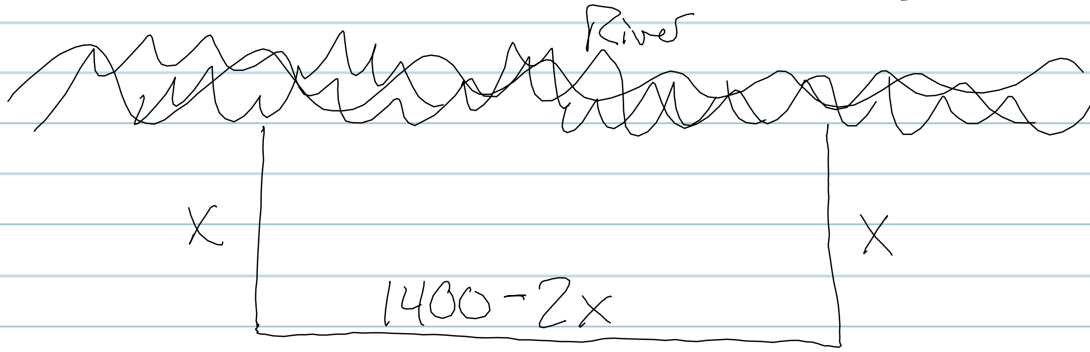
$V(0) = 0$

$V(2) = (4 \cdot 2)(6-2)^2 = 8(4^2) = 16 \cdot 8 = 128$ ← max

$V(6) = 4(6)(6-6) = 0$

$\Rightarrow x = 2in$ maximizes the volume of the box to $128 in^3$

(c) A campground owner has 1400m of fencing. He wants to enclose a rectangular field bordering a river w/ no fencing needed along the river. Let x represent the width of the field.



First we need the length of the field: $\boxed{L = 1400-2x}$

Now, we want to maximize Area, so we need a function to represent the area of the field

$$A(x) = Lw = x(1400-2x) = 1400x - 2x^2$$

Lets Maximize!

$$A'(x) = 1400 - 4x = 4(350-x)$$

So one critical value $x_c = 350$

Is it a relative max?

$f'(349) = 4(350-349) > 0$ $\Rightarrow$ relative max!
$f'(351) = 4(350-351) < 0$

Domain? $x=0$ to $1400-2x \geq 0 \Rightarrow x \leq \frac{1400}{2} = 700$

Check endpts

$A(0) = 0$

$A(700) = 700(1400-2\cdot 700) = 0$

Thus $x = 350$m leads to the maximum area of $A(350) = 245{,}000 m^2$

§6.4: Implicit Differentiation

There are implicit functions and explicit functions...

For an explicit function, y, as a variable of x, we know the formula for y, directly in terms of x:

e.g. $y(x) = 3x+2$, $y(x) = 3x^2+4x+7$; $y(x) = -x^3+2$

What if we don't know the formula for "y in terms of x"

e.g. what is y in the equation

$$y^5 + 8y^3 + 6y^2x^2 + 2xy^3 + 6 = 0 \text{ ?}$$

Here y is said to be given "implicitly in terms of x"

Implicit Differentiation:

To find $\frac{dy}{dx}$ $(= y'(x))$ for an equation containing x and y

(1) Differentiate both sides with respect to x, keeping in mind $y = y(x)$ (i.e. chain rule!)

(2) By the power of algebra, solve for $\frac{dy}{dx}$

e.g. $8x^2 - 10xy + 3y^2 = 26$; Find dy/dx

Take the derivative of both sides but every time we have a y term we must multiply by dy/dx

$$\Rightarrow 16x - 10\left[(1)y + x\frac{dy}{dx}\right] + 6y\left[\frac{dy}{dx}\right] = \frac{d}{dx}[26] = 0$$

$$\Rightarrow (-10x+6y)\frac{dy}{dx} = 10y - 16x$$

$$\Rightarrow \frac{dy}{dx} = \frac{10y-16x}{6y-10x} = \frac{-\cancel{2}(-5y+8x)}{-\cancel{2}(-3y+5x)} = \boxed{\frac{8x-5y}{5x-3y}}$$

e.g.) $4\sqrt{x} - 8\sqrt{y} = 6y^{3/2}$; Find $\frac{dy}{dx}$

Again, take the derivative of both sides:

$$4\left(\frac{1}{2}x^{-1/2}\right) - 8\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dx} = 6\left(\frac{3}{2}y^{1/2}\right)\frac{dy}{dx}$$

rearrange

$$\Rightarrow 9y^{1/2}\frac{dy}{dx} + 4y^{-1/2}\frac{dy}{dx} = \left(9y^{1/2} + 4y^{-1/2}\right)\frac{dy}{dx} = 2x^{-1/2}$$

algebra

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{x}} \cdot \frac{1}{\left(9y^{1/2} + 4y^{-1/2}\right)} = \frac{2}{\sqrt{x}}\left[\frac{1}{y^{-1/2}(9y+4)}\right] = \boxed{\frac{2\sqrt{y}}{\sqrt{x}(9y+4)}}$$

e.g.) $e^{x^2y} = 5x + 4y + 2$; Find dy/dx

Take $\frac{d}{dx}$ both sides

$$\Rightarrow e^{x^2y}\left[x^2\frac{dy}{dx} + 2xy\right] = 5 + 4\frac{dy}{dx}$$

$$\Rightarrow \left[x^2e^{x^2y} - 4\right]\frac{dy}{dx} = 5 - 2xye^{x^2y} \Rightarrow \boxed{\frac{dy}{dx} = \frac{5 - 2xye^{x^2y}}{x^2e^{x^2y} - 4}}$$

e.g.) $y\ln x + 2 = x^{3/2}y^{5/2}$; Find dy/dx

take derivatives

$$\Rightarrow y \cdot \frac{1}{x} + \frac{dy}{dx}\ln x = x^{3/2} \cdot \frac{5}{2}y^{3/2}\frac{dy}{dx} + \frac{3}{2}x^{1/2}y^{5/2}$$

$$\Rightarrow \left[\ln x - \frac{5}{2}y^{3/2}x^{3/2}\right]\frac{dy}{dx} = \frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x}$$

algebra

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{2x}\left[3x^{3/2}y^{5/2} - 2y\right]}{\left[\ln x - \frac{5}{2}y^{3/2}x^{3/2}\right]} = \boxed{\frac{3x^{3/2}y^{5/2} - 2y}{x\left(2\ln x - 5y^{3/2}x^{3/2}\right)}}$$

e.g.) Eqn of tangent line at (1,1) for the equation

$$8x^2 - 10xy + 3y^2 = 26$$

Previously we found the slope of the tangent line, $\frac{dy}{dx} = m$

$$y'(x,y) = \frac{dy}{dx} = \frac{8x-5y}{5x-3y}$$

Now plug in $x=1, y=1$ to find the slope

$$m = y'(1,1) = \frac{8(1)-5(1)}{5(1)-3(1)} = \frac{8-5}{5-3} = \frac{3}{2}$$

Now we point slope form to find the equation of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 1 = \frac{3}{2}(x-1) = \frac{3}{2}x - \frac{3}{2}$$

$$\Rightarrow y = \frac{3}{2}x - \frac{3}{2} + \frac{2}{2}$$

$$\Rightarrow \boxed{y = \frac{3}{2}x - \frac{1}{2}}$$

§ 6.5 : Related Rates

- Here consider two functions (dependent variables) varying with time, i.e. $x(t)$ and $y(t)$ (so $x'(t) = \frac{dx}{dt}$ and $y'(t) = \frac{dy}{dt}$)
- If x and y are constrained by an equation, then so too will their derivatives (rates)

First let's consider purely algebraic examples:

e.g. For each assume x and y are functions of t $(x(t), y(t))$
Use implicit differentiation to find $\frac{dy}{dt}$

(A) Evaluate $\frac{dy}{dt}$ when $x=3$, $y=-1$ and $\frac{dx}{dt}=2$ given the equation

$$8y^3 + x^2 = 1$$

First take the derivative of both sides of the equation with respect to t

$$\Rightarrow 8\left(3y^2 \frac{dy}{dt}\right) + 2x\left(\frac{dx}{dt}\right) = 0$$; Now solve for $\frac{dy}{dt}$

$$\Rightarrow \frac{dy}{dt} = \frac{-2x\left(\frac{dx}{dt}\right)}{(8\cdot 3)y^2}$$; Now remember $\frac{dy}{dt} = y'(x, y, \frac{dx}{dt})$
(y' is a function of x, y, and $\frac{dx}{dt}$)

Plug in each of these values:

$$y'\left(x=3, y=-1, \frac{dx}{dt}=2\right) = \frac{-2(3)(2)}{(8\cdot 3)(-1)^2} = \frac{-4}{8} = \boxed{\frac{-1}{2}}$$

(8) Evaluate $\frac{dy}{dt}$ when $x=1$, $y=0$, and $\frac{dx}{dt}=5$ given the equation

$$y\ln x + xe^y = 1$$

First take the derivative with respect to t of each side of the equation (we need the product rule)

$$\Rightarrow y\left(\frac{1}{x}\cdot\frac{dx}{dt}\right) + \left(1\cdot\frac{dy}{dt}\right)\ln x + x\left(e^y\frac{dy}{dt}\right) + \left(1\cdot\frac{dx}{dt}\right)e^y = 0$$

rearrange

$$\Rightarrow \frac{dy}{dt}\left(\ln x + xe^y\right) = \left(\frac{-y}{x} - e^y\right)\frac{dx}{dt}$$

Now solve for $\frac{dy}{dt}$

$$\frac{dy}{dt} = \frac{\left(\frac{-y}{x} - e^y\right)\frac{dx}{dt}}{\ln x + xe^y}$$

: Again $\frac{dy}{dt}$ is a function of $x, y, \frac{dx}{dt}$ $\left(\frac{dy}{dt} = y'\left(x, y, \frac{dx}{dt}\right)\right)$

Finally, plug in these values to find $\frac{dy}{dt}$ at this given point

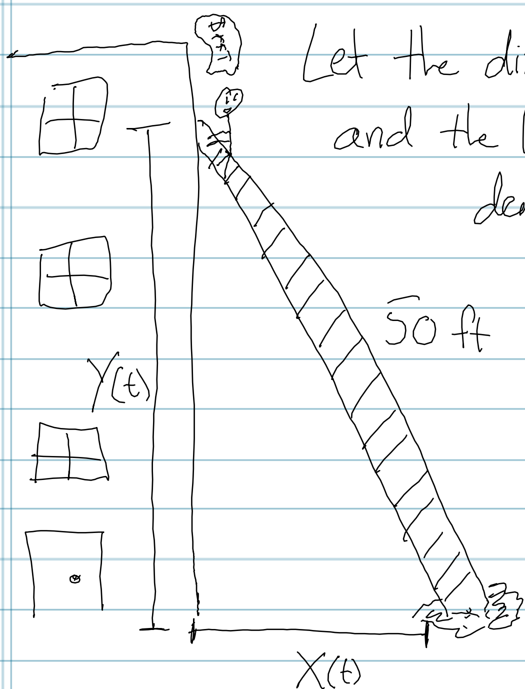
$$y'\left(x=1, y=0, \frac{dx}{dt}=5\right) = \frac{\left(\frac{-0}{1} - e^0\right)(5)}{(\ln 1) + (1e^0)} = \frac{(0-1)5}{0+1(1)} = \frac{-5}{1} = -5$$

Now, we consider some real-world examples of the application of related rates.

Solving Related Rates Problems

1. Identify all given quantities and make a sketch if possible
2. Write an equation relating the variables
3. Use implicit differentiation to solve for the unknown rate of change in terms of the given quantities.

e.g.) A 50-ft ladder leans against a large building. The base of the ladder rests upon an oil slick, and it slips out at a rate of 3ft per minute. Find the speed that an unfortunate occupant at the top of the ladder would fall at the instant the ladder's base is 30ft from the edge of the building.



Let the distance of from the ladder to the building be x ft and the height of the ladder against the building be denoted y ft. The speed the ladder is falling, will be the rate of change of the height $\frac{d}{dt}Y(t) = \frac{dy}{dt}$

First we need an equation relating x, y and the length of the ladder. (Pythagorus ?)

$\Rightarrow$ $x^2 + y^2 = 50^2$. Now use implicit differentiation to find $\frac{dy}{dt}$ (i.e. take the derivative of both sides of the eqn wrt. t.)

$\xRightarrow{\frac{d}{dt}}$ $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \frac{d}{dt}\left[50^2\right] = 0$

$\xRightarrow{\text{solve for } \frac{dy}{dt}}$ $\frac{dy}{dt} = \frac{-2x\left(\frac{dx}{dt}\right)}{2y} = \frac{-x\left(\frac{dx}{dt}\right)}{y}$

Now we need to consider the problem to find values for x, y, and $\frac{dx}{dt}$

we are finding $\frac{dy}{dt}$ when the ladder is 30ft from the base of the building

$\Rightarrow$ $x = 30$

We also know the base of the ladder is slipping at a rate of 3ft per minute

$\Rightarrow \frac{dx}{dt} = 3 \text{ ft/min}$

Finally, we need y = height of the ladder. Use our first equation:

$x^2 + y^2 = 50^2 \Rightarrow 30^2 + y^2 = 50^2$

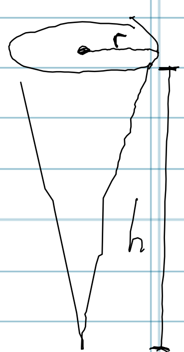
$\Rightarrow y^2 = 50^2 - 30^2 = 2500 - 900 = 1600$

$\Rightarrow y = 40 \text{ft}$

Plug these in to the equation we found for $\frac{dy}{dt} = y'(x, y, \frac{dx}{dt}) = \frac{-x}{y}\left(\frac{dx}{dt}\right)$

$y'(x=30, y=40, \frac{dx}{dt}=3) = \frac{-30}{40}(3) = -\frac{3}{4}(3) = -\frac{9}{4} = -2\frac{1}{4} \text{ ft/min}$

e.g.) A cone shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of $\frac{1}{2}$cm per hour, while the length is increasing by 1 cm/hr. If the icicle is currently 4cm in radius and 20cm long, is the volume of the icicle increasing or decreasing and at what rate?



The Volume of a cone is $V = \frac{1}{3}\pi r^2 h$, (as displayed)

Here, $r = r(t)$, $h = h(t)$, and $V = V(t)$ where t = time in hrs

Use implicit differentiation to find $\frac{dV}{dt}$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\left[\left(2r \cdot \frac{dr}{dt}\right)h + r^2\left(1 \cdot \frac{dh}{dt}\right)\right]$$

Now the radius is decreasing by at 1/2 cm/hr and length is increasing at 1 cm/hr

$\Rightarrow$ We are interested in the rate of change when $r = 4$cm, $h = 20$cm

$$\frac{dr}{dt} = -\frac{1}{2} \text{ cm/hr} \; ; \; \frac{dh}{dt} = 1 \text{ cm/hr}$$

Now all we have to do is plug in these values to find the change in volume at this time:

$$\frac{dV}{dt} = \frac{1}{3}\pi \left[2(4\text{cm})\underbrace{(\ \ )}_{-\frac{1}{2}\text{cm/hr}}(20\text{cm}) + (4\text{cm})^2 (1 \text{ cm/hr}) \right]$$

$$= \frac{\pi}{3}\left[-80 \text{ cm}^3/\text{hr} + 16 \text{cm}^3/\text{hr} \right]$$

$$= \frac{\pi}{3}\left[-64 \text{cm}^3/\text{hr} \right] \approx -67.02 \text{ cm}^3/\text{hr}$$

So volume is decreasing, (as expected)