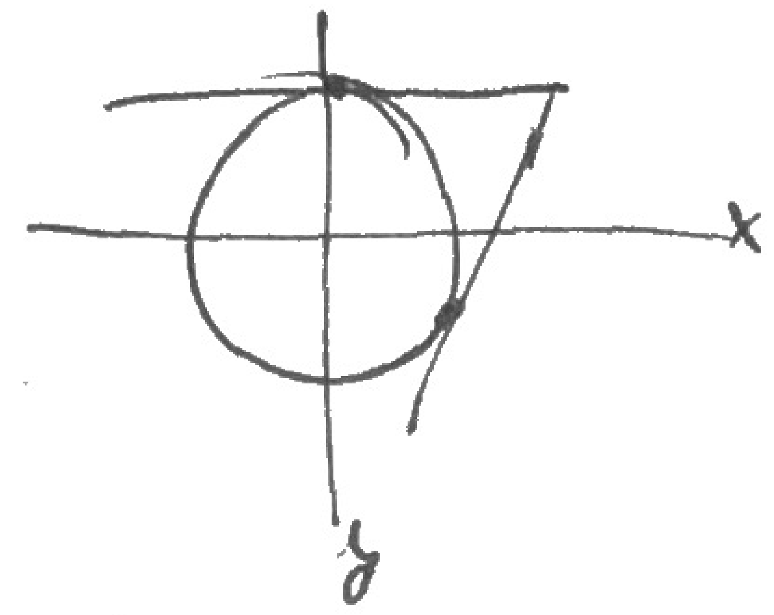
Section 3.4: Definition of the derivative

Previously, the formula

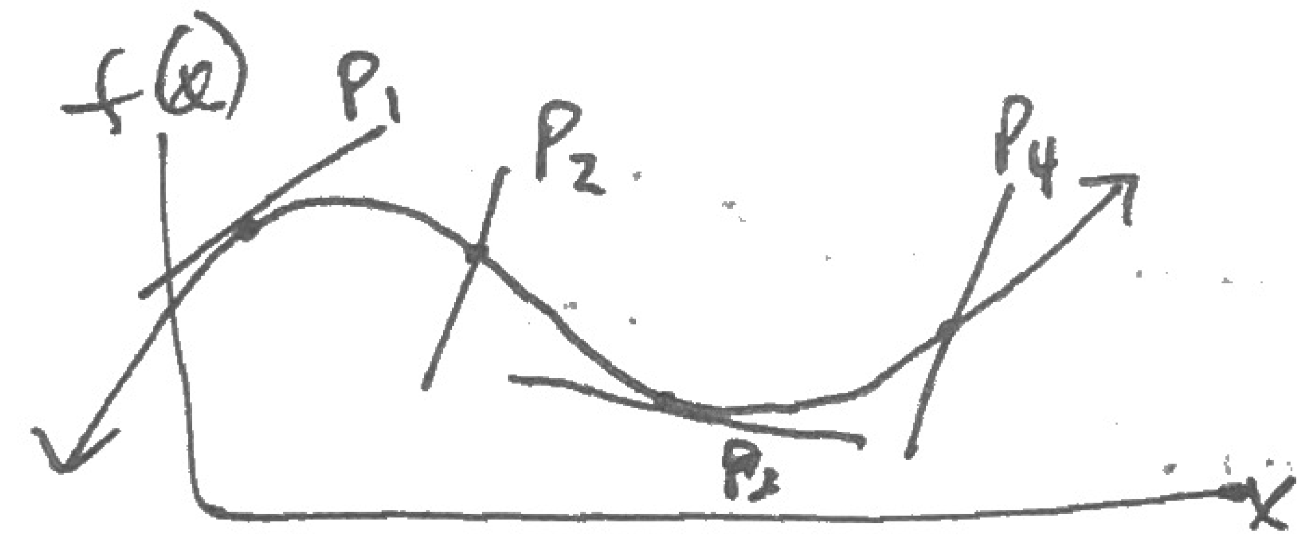
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

was used to compute the inst. rate of change.

Geometrically, a tangent to a circle is ?

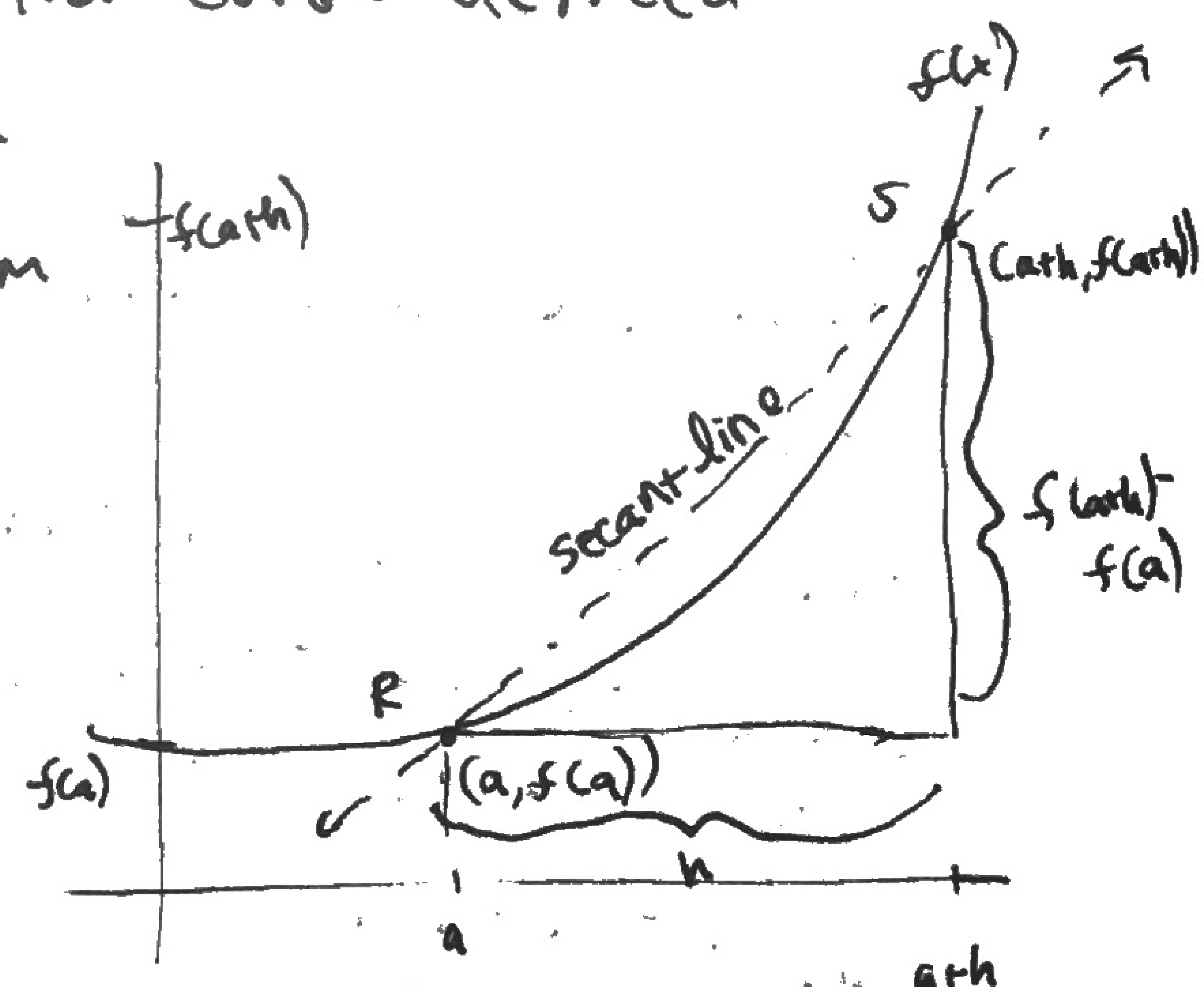


the tangent should "touch" the curve, but not at more than one point, and should indicate the direction.



which are tangent lines?

To define the slope of the line tangent to the curve, let R be a point on the curve defined by $(a, f(a))$. Choose S at the value $(a+h, f(a+h))$. The line from R to S is a secant line, with slope? From defn:



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

As h approaches 0, S will slide down the curve, getting closer and closer to R.

Defn The tangent line of the graph $y=f(x)$ at the point $(a, f(a))$ is the line through this pt having slope $\lim_{h\to 0} \frac{f(a+h)-f(a)}{a+h-a}$, provided this limit exists. (if it doesn't, no tangent line).

Familiar? of course. Slope of the curve is the same as the instantaneous rate of change of y with respect to x. Direction of curve at that point.

Ex 1 Consider $f(x) = x^2+2$.

(a) find the slope and eq of the secant line through points where $x=-1$ and $x=2$.

Sol'n

slope of secant line $= \frac{\Delta y}{\Delta x} = \frac{f(2)-f(-1)}{2 - -1} = \frac{6-3}{3} = 1$

use point-slope form:

$$y - y_1 = m(x-x_1)$$

$$y - f(2) = 1(x-2) = 1(x-2) = x-6$$

$$\Rightarrow y-6 = x-2 \to y = x+4.$$

(b) find the slope and equation of the tangent line at $x=-1$.

Soln use the def'n: slope of tangent $= \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a}$

$$= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{-1+h-(-1)} = \lim_{h\to 0}\frac{(-1+h)^2+2-((-1)^2+2)}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{1}-2h+h^2+\cancel{2}-\cancel{1}-\cancel{2}}{h} = \lim_{h\to 0}\frac{-2h+h^2}{h} = \lim_{h\to 0} -2+h = -2.$$

Eq: use point slope form.

Defn The derivative of the function f at x is defined as

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \quad \text{(provided it exists)}.$$

↑ derivative of f

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existence ⇒ function is differentiable

(1) $f'(x)$ is the instantaneous rate of change.

*interpretation depends on what f is!

* if f is position, f' is velocity.
* velocity → acceleration
* bacteria count → growth rate

(2) $f'(x)$ is the slope of the graph $f(x)$ at any point x. (tangent line)

Defn The derivative of a function f at x can be written as

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}. \quad \text{(if it exists).}$$

Ex 4] Let $f(x) = x^2$. (a) Find the derivative.

Soln use either definition.

(1) $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$

$f(x) = x^2$

$$\Rightarrow f'(x) = \lim \frac{f(x+h) - f(x)}{x+h - x} = \lim \frac{x^2 + 2xh + h^2 - (x^2)}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x.$$

(2) $f(6)=6^2$, $f(x)=x^2$

$$\Rightarrow \lim_{x\to 6}\frac{f(6)-f(x)}{6-x} = \lim_{6\to x}\frac{6^2-x^2}{6-x} = \lim_{6\to x}\frac{(6+x)\cancel{(6-x)}}{\cancel{6-x}} = \lim_{6\to x} 6+x = 2x$$

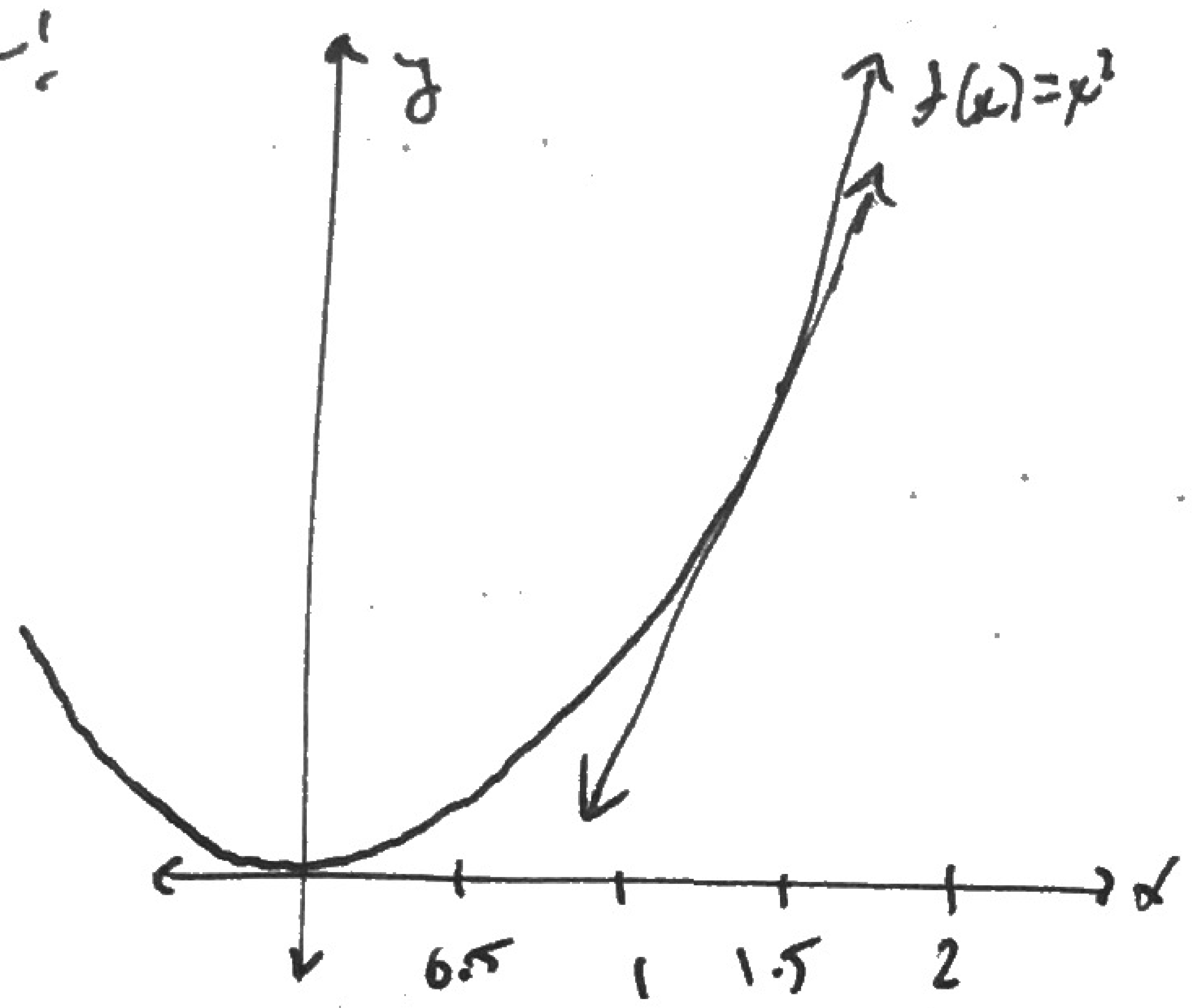
* this method was easier here, but won't always be!

(b) Calculate and interperet $f'(3)$.

(1) of course, $f'(x)=2x \Rightarrow f'(3)=6$.

slope is 6!

(2) graph it!



slope of secant line $= \dfrac{f(2.001)-f(2)}{2.001-2} = \dfrac{(2.001)^2-2^2}{.001} = \dfrac{4.004001-4}{.001}$

$= 4.001$

$$\frac{f(3.001)-f(3)}{3.001-3} = \frac{(3.001)^2-9}{.001} = \frac{9.006001-9}{.001} = \frac{.006001}{.001}$$

$= 6.001$

Finding $f'(x)$: step-by-step

(1) find $f(x+h)$.

(2) Find and simplify $f(x+h)-f(x)$

(3) divide by h to get $\dfrac{f(x+h)-f(x)}{h}$

(4) Plug in 0, if you can!

Equivalent expressions: x_2-x_1, $6-a$, Δx, h.

Find $f(x)=2x^3+9x$. Find $f'(x)$, $f'(2)$, $f'(-3)$

Try $f(x) = 2x^3 + 4x$. find $f'(x)$, $f'(2)$, $f'(-3)$.

<u>Soln</u> go step by step!

(1) $f(x+h) = 2(x+h)^3 + 4(x+h)$

$= 2(x+h)(x^2+2hx+h^2) + 4x + 4h$

$\cancel{= 2(x^3+2hx+xh^2+hx^2+2h^2+h^3)+4x+4h}$

$\cancel{= 2x^3+4hx+4h^2 \ldots}$

$\cancel{= 2(x^3+2hx+2xh^2+2h^2+h^3)+4x+4h}$

$\cancel{= 2x^3+4hx+4h^2x+4h^2+2h^3+4x+4h}$

$= 2(x^3+2hx^2+h^2x+hx^2+2h^2x+h^3) + 4x + 4h$

$= 2(x^3+3hx^2+3h^2x+h^3) + 4x + 4h$

$= 2x^3 + 6hx^2 + 6h^2x + 2h^3 + 4x + 4h$

(2) $f(x+h) - f(x) = \cancel{2x^3} + 6hx^2 + 6h^2x + 2h^3 + \cancel{4x} + 4h - (\cancel{2x^3} + \cancel{4x})$

$= 6hx^2 + 6h^2x + 2h^3 + 4h$

(3) $\dfrac{f(x+h) - f(x)}{h} = 6x^2 + 6hx + 2h^2 + 4$

(4) $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} 6x^2 + 6hx + 2h^2 + 4 = 6x^2 + 4$.

$f'(x) = 6x^2 + 4$.

$f'(2) = 6(2)^2 + 4 = 6 \cdot 4 + 4 = 28$

$f'(-3) = 6(-3)^2 + 4 = 6 \cdot 9 + 4 = 58$

EX) Let $f(x) = 4/x$. Find $f'(x)$.

(1) $f(x+h) = 4/(x+h)$

(2) $f(x+h) - f(x) = \frac{4}{x+h} - \frac{4}{x}$

$$= \frac{4x}{(x+h)(x)} - \frac{4(x+h)}{(x)(x+h)} = \frac{-4h}{x(x+h)}$$

(3) $\frac{f(x+h)-f(x)}{h} = \frac{-4}{x(x+h)}$

(4) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-4}{x(x+h)} = \frac{-4}{x(x+0)} = \frac{-4}{x^2}$

(done!).

EX) I switched Finn's dog food to pork for the winter, and after x weeks of the new food, his weight is given by $w(x) = \sqrt{x} + 40$, for $0 \le x \le 6$.
Find the rate of change of Finn's weight after x weeks.

SOLN) step 1: $w(x+h) = \sqrt{x+h} + 40$

2: $w(x+h) - w(x) = \sqrt{x+h} + 40 - \sqrt{x} - 40$

$$= \sqrt{x+h} - \sqrt{x}$$

3: $\frac{w(x+h) - w(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$ rationalize!

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{\sqrt{x+h}^2 - \sqrt{x}^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Step 4: $$w'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

after 4 weeks, Finn is gaining

$$w'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4} \text{ lbs/week.}$$

and weighs $w(4) = \sqrt{4} + 40 = 2 + 40 = 42$ lbs.

Equation of the tangent line

The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the eq

$$y - f(x_1) = f'(x_1)(x - x_1)$$

provided $f'(x_1)$ exists.

Ex) Find the tangent line to $4/x$ at $x = 2$.

soln) know $f'(x) = -4/x^2$, so

$$f'(2) = \frac{-4}{2^2} = \frac{-4}{4} = -1 \quad \text{and } f(2) = \frac{4}{2} = 2$$

$$\Rightarrow \quad y - 2 = -1(x - 2)$$

$$= -x + 2$$

$$y = -x + 4. \quad \checkmark$$

→ show figure 43.

Existence of the Derivative

The derivative exists when a function f satisfies all of the following conditions at a point.

1. f is continuous.
2. f is smooth and
3. f does not have a vertical tangent line.

The derivative does not exist when any of the following conditions are true for a function at a point.

1. f is discontinuous.
2. f has a sharp corner.
3. f has a vertical tangent line.