§4.2: Derivatives of Products and Quotients

We know that the derivative of the sum is the sum of the derivatives. What about the derivative of a product or quotient?

Is it just $\frac{d}{dx}[f(x)\cdot g(x)] \stackrel{?}{=} f'(x)\cdot g'(x)$? Let's investigate.

Suppose $f(x) = 3x+2$ and $g(x) = 4x^3$

Then (power rule) $\frac{d}{dx}(f(x)) = 3$ and $\frac{d}{dx}g(x) = 12x^2$ but:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(12x^4 + 8x^3) = 48x^3 + 24x^2 \neq 3\cdot 12x^2 = f'(x)g'(x)$$

So, we must devise a formula for the derivative of a product!

For some general function $f(x) = u(x)v(x)$, lets evaluate $f'(x)$ using the formal definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

nearby trick: add and subtract $u(x+h)v(x)$ from the numerator

$$= \lim_{h\to 0}\frac{u(x+h)v(x+h) + \left[u(x+h)v(x) - u(x+h)v(x)\right] - u(x)v(x)}{h}$$

rearrange, factor

$$= \lim_{h\to 0}\frac{u(x+h)[v(x+h)-v(x)]}{h} + \lim_{h\to 0}\frac{v(x)[u(x+h)-u(x)]}{h}$$

$$= u(x)v'(x) + v(x)u'(x)$$

This gives us a formula for the derivative of the product, AKA The Product Rule!

The Product Rule: Let $f(x) = u(x)v(x)$ s.t. u and v are differentiable (i.e. $u'(x)$, $v'(x)$ exists). Then

$$\boxed{f'(x) = u(x)v'(x) + v(x)u'(x)}$$

e.g.1 Find the derivative with the Product rule

(A) $f(x) = \underset{u}{(3x+2)}\underset{v}{(4x^3)} \Rightarrow f'(x) = \underset{u}{(3x+2)}\underset{v'}{(3\cdot 4x^{3-1})} + \underset{u'}{(3x^{1-1})}\underset{v}{(4x^3)} = (3x+2)(12x^2) + 12x^3$

$= 36x^3 + 12x^3 + 24x^2$

$= \boxed{48x^3 + 24x^2}$

(B) $y = (x+1)(\sqrt{x}+2) = (x+1)(x^{1/2}+2) \Rightarrow \frac{dy}{dx} = (x+1)\left[\frac{1}{2}x^{\frac{1}{2}-1} + 0\right] + [1+0](\sqrt{x}+2) = (x+1)\left(\frac{1}{2}x^{-1/2}\right) + (\sqrt{x}+2)$

$= \frac{1}{2}x^{\frac{1}{2}-\frac{1}{2}} + \frac{1}{2}x^{-1/2} + \sqrt{x} + 2 = \boxed{\frac{3}{2}x^{+1/2} + \frac{1}{2}x^{-1/2} + 2}$ (combine)

$= \frac{3\sqrt{x}}{2} + 2 + \frac{1}{2\sqrt{x}}$

(C) $P(y) = \left(\frac{1}{y} + \frac{1}{y^2}\right)\left(\frac{2}{y^3} - \frac{5}{y^4}\right) = (y^{-1}+y^{-2})(2y^{-3} - 5y^{-4}) \Rightarrow P'(y) = (y^{-1}+y^{-2})(-6y^{-4} + 20y^{-5}) + (-y^{-2} - 2y^{-3})(2y^{-3} - 5y^{-4})$

$= -6y^{-5} + 20y^{-6} - 6y^{-6} + 20y^{-7} - 2y^{-5} + 5y^{-6} - 4y^{-6} + 10y^{-7}$ ($5y^{-4}$)

rearrange, combine

$\Rightarrow \boxed{P'(y) = -8y^{-5} + 15y^{-6} + 30y^{-7}}$

How about the derivative of the quotient of functions? For that, we have:

The Quotient Rule:

If $f(x) = \frac{u(x)}{v(x)}$ and u', v' exist with $v(x) \neq 0$, then:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Or, you can remember the jingle 🎵

Low D High − High D Low all over LowLow

$$\left\{\frac{(v(x))(u'(x)) - u(x)v'(x)}{v(x)v(x)}\right\}$$

e.g. Find the derivative with the quotient rule

(A) $f(t) = \dfrac{4t^2+11}{t^2+3}$

$u(x) = 4t^2+11$

Here we can use the quotient rule with: $v(x) = t^2+3$

By Power R. le: $u'(x) = 8t$; $v'(x) = 2t$

Plug into our quotient rule formula:

$$f'(t) = \frac{u'v - uv'}{v^2} = \frac{(8t)(t^2+3) - [(4t^2+11)(2t)]}{(t^2+3)^2} = \frac{8t^3+24t - 8t^3 - 22t}{(t^2+3)^2}$$

$$= \boxed{2t/(t^2+3)^2}$$

* Notice, in general you can usually leave the denominator in factored form

(B) $f(x) = \dfrac{(3x^2+1)(2x-1)}{5x+4}$

Here let $u(x) = (3x^2+1)(2x-1)$ and $v(x) = 5x+4$, $v'(x) = 5$

To find $u'(x)$, use the product rule

$$u'(x) = (3x^2+1)[2] + [6x](2x-1) = 6x^2+2+12x^2-6x$$
$$= 18x^2-6x+2$$

Now, use the quotient rule:

$$f'(x) = \frac{(18x^2-6x+2)(5x+4) - (3x^2+1)(2x-1)(5)}{[5x+4]^2}$$

$$= \frac{90x^3+72x^2-30x^2+8 \overset{(-24x+10x)}{\swarrow} - [(6x^3-3x^2+2x-1)5]}{(5x+4)^2}$$

$$= \frac{90x^3+72x^2-30x^2+8 \overset{(-24x+10x)}{\swarrow} - 30x^3+15x^2-10x+5}{(5x+4)^2}$$

$$= \frac{60x^3+57x^2-24x+13}{(5x+4)^2}$$

Average Cost: Suppose $y = C(x)$ gives the total cost to manufacture x items

Then the average cost per item is $\bar{y} = \bar{C}(x) = \frac{C(x)}{x}$

i.e. The cost of x items over the total # of items

Marginal Average Cost

This is simply the derivative (i.e. rate of change) of the average cost function

So if the average cost is given by: $\bar{C}(x) = C(x)/x$, then

Marginal Avg Cost $\equiv \bar{C}'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right)$

A company may want to make the average cost as small as possible:

e.g. Suppose the ~~average~~ total cost (in 100's of dollars) to produce x units of skis is $C(x) = \frac{3x+2}{x+4}$; Find the marginal average cost function.

Before we can find the marginal average cost, we need the average cost function.

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{3x+2}{x+4} \cdot \frac{1}{x} = \frac{3x+2}{x^2+4x}$$

To find the marginal average cost we need to use the quotient rule:

$$\bar{C}'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{(3)(x^2+4x) - (3x+2)(2x+4)}{(x^2+4x)^2}$$

$$= \frac{3x^2+12x - [6x^2+12x+4x+8]}{(x^2+4x)^2} = \frac{-3x^2-4x-8}{(x^2+4x)^2}$$

One may want to minimize the average cost. This generally occurs when the marginal average cost (its derivative ∇) is 0. This "minimization" technique will be covered in more detail in the next chapters.

§1.3: The Chain Rule

The chain rule describes how one takes the derivative of a composition of functions. Lets review:

Composite Function

Let f and g be functions. The composite function, $g \circ f(x)$ is the function whose values are given by $g(f(x))$, $\forall x$ in the domain of f ST $f(x)$ is in the domain of g

e.g. Let $f(x) = 2x-1$ and $g(x) = \sqrt{3x+5}$. Find:

(A) $g \circ f(x) = g(f(x)) = \sqrt{3(2x-1)+5} = \sqrt{6x-3+5} = \sqrt{6x+2}$

(B) $f \circ g(x) = 2\left[\sqrt{3x+5}\right] - 1$

(C) $g \circ f(4)$:

$f(4) = 2 \cdot 4 - 1 = 7$; $g(7) = \sqrt{3 \cdot 7 + 5} = \sqrt{21+5} = \sqrt{26}$

OR

$g \circ f(4) = \sqrt{3[2 \cdot 4 - 1] + 5} = \sqrt{3[7]+5} = \sqrt{21+5} = \sqrt{26}$

e.g. Write each as the composition of 2 functions f and g so that $h(x) = f \circ g(x)$

(A) $h(x) = 2(4x+1)^2 + 5(4x+1)$

Let $g(x) = 4x+1 \Rightarrow h(x) = 2[g(x)]^2 + 5[g(x)]$

Now let $f(x) = 2x^2 + 5x$

$\Rightarrow h(x) = f \circ g(x) = f(4x+1) = 2(4x+1)^2 + 5(4x+1)$

(B) $h(x) = f \circ g(x) = \sqrt{1-x^2}$: Here we are taking the square root of a quadratic

$\Rightarrow g(x) = (1-x^2)$; $f(x) = \sqrt{x}$

$\Rightarrow h(x) = f \circ g(x) = f(1-x^2) = \sqrt{1-x^2}$

The reason for the previous exercise is to identify a function that is expressible as a composition of others, so that we may take the derivative in terms of the composition

The Chain Rule

If $h(x) = f \circ g(x) = f(g(x))$, and both $f'(x), g'(x)$ exist, then we may find the derivative $h'(x)$ as

$$h'(x) = g'(x) \cdot f'(g(x))$$

Chain Rule (Alternate Form)

If y is a function of u, $y = f(u)$ and if u is a function of x, say $u = g(x)$ then $y = f(u) = f(g(x))$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Here we can remember the chain rule by pretending that $\frac{dy}{du}$ and $\frac{du}{dx}$ are fractions with du "cancelling out"

e.g. Find the derivative using the chain rule. where needed

(a) $D_x\left[8x^4 - 5x^2 + 1\right]^4$ If we think of $f(x)$ as $f(x) = x^4$, we can use the chain rule instead of foiling 4 times!

$$\Rightarrow D_x\left[8x^4 - 5x^2 + 1\right]^4 = 4\left[8x^4 - 5x^2 + 1\right]^{4-1} \cdot \frac{d}{dx}\left[8x^4 - 5x^2 + 1\right]$$

$$= 4\left[8x^4 - 5x^2 + 1\right]^3 \left[32x^3 - 10x\right]$$

(b) $\psi(t) = -6t\left(5t^4 - 1\right)^4$ Now we need the product rule and chain rule!

$$\psi'(t) = -6t\left[4\left(5t^4 - 1\right)^3 (20t^3)\right] + (-6)\left(5t^4 - 1\right)^4 = -6\left[\left(5t^4 - 1\right)^3 (80t^4)\right] + (-6)\left(5t^4 - 1\right)^4$$

$$= -6\left(5t^4 - 1\right)^3 \left[80t^4 + \left(5t^4 - 1\right)\right] = -6\left(5t^4 - 1\right)^3 \left[85t^4 - 1\right]$$

(c) $\xi(t) = \dfrac{(5t-6)^4}{3t^2+4}$ Here, we need the quotient rule and chain rule

$$\xi'(t) = \frac{(3t^2+4)\left[4(5t-6)^3(5)\right] - (5t-6)^4[6t]}{[3t^2+4]^2}$$ Now factor the numerator

$$\Rightarrow \xi'(t) = \frac{2(5t-6)^3\left[10(3t^2+4) - (5t-6)(3t)\right]}{[3t^2+4]^2} = \frac{2(5t-6)^3[30t^2+40-15t^2+18t]}{[3t^2+4]^2}$$

$$= \frac{2(5t-6)^3[15t^2+18t+40]}{[3t^2+4]^2}$$

Using the chain rule, we can easily prove the quotient rule

Pf of Quotient Rule: Suppose $f(x) = \dfrac{u(x)}{v(x)} = u(x)[v(x)]^{-1}$. Then, by the product and chain rule:

$$f'(x) = u'(x)[v(x)]^{-1} + u(x)\left[-1(v(x))^{-2}v'(x)\right] = \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{[v(x)]^2}$$

common denominator $[v(x)]^2$ $\Rightarrow$ $\boxed{f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}}$

e.g. Find the derivative using the product and chain rule (Not quotient rule!)

$f(x) = \dfrac{2x-1}{4x+3} = (2x-1)(4x+3)^{-1}$, then

$$f'(x) = (2x-1)\left[-1(4x+3)^{-2}(4)\right] + 2(4x+3)^{-1} = \frac{-4(2x-1)}{(4x+3)^2} + \frac{2}{(4x+3)}$$

$$= \frac{(-8x+4) + 2(4x+3)}{(4x+3)^2} = \frac{-8x+8x+4+6}{(4x+3)^2} = \boxed{\frac{10}{(4x+3)^2}}$$

§4.4: Derivatives of Exponential Functions

What is the derivative of the exponential function: $f(x)=e^x$?

Derivative of e^x:

$$\boxed{\frac{d}{dx}e^x = e^x}$$

So what is the derivative of a^x, for any $a \in \mathbb{R}$ with $a>0, a\neq 1$

e.g.) Observe

$a^x = e^{\ln a^x} = e^{x\ln a}$, then by the chain rule:

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{x\ln a}\right] = (\ln a)e^{x\ln a} = (\ln a)e^{\ln a^x} = (\ln a)a^x$$

Derivative of a^x; $a>0, a\neq 1$:

$$\boxed{\frac{d}{dx}[a^x] = (\ln a)a^x}$$

Derivative of $a^{g(x)}$ and $e^{g(x)}$: (By the chain rule)

$$\frac{d}{dx}\left[a^{g(x)}\right] = (\ln a)\left(a^{g(x)}\right)\cdot g'(x)$$

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\cdot g'(x)$$

e.g. Find the derivative:

(A) $y = -8e^{3x} \Rightarrow \frac{dy}{dx} = (-8)(e^{3x}) \cdot \frac{d}{dx}(3x) = (-8)e^{3x} \cdot 3 = -24e^{3x}$

(B) $s = 4^{-5t+2} \Rightarrow \frac{ds}{dt} = \left[(\ln 4)(4^{-5t+2})\right] \cdot -5 = (-5\ln 4)\,4^{-5t+2}$

(C) $\psi = -3e^{3x^2+5} \Rightarrow \frac{d\psi}{dx} = \left(-3e^{3x^2+5}\right) \cdot \frac{d}{dx}(3x^2+5) = (-18x)e^{3x^2+5}$

(D) $y = \underbrace{(3x^3 - 4x)}_{(u)}\underbrace{e^{-5x}}_{(v)} \overset{\text{product rule}}{\Rightarrow} y' = uv' + u'v = (3x^3-4x)(-5e^{-5x}) + (9x^2-4)e^{-5x}$

$= e^{-5x}\left[-15x^3 + 20x + 9x^2 - 4\right]$

(E) $f(t) = (e^{t^2} + 5t)^3 \Rightarrow f'(t) = 3(e^{t^2}+5t)^2 \cdot \frac{d}{dt}\left[e^{t^2}+5t\right]$

$= \boxed{3(e^{t^2}+5t)^2 \cdot (2te^{t^2}+5)}$

(F) $y = \frac{x^2}{e^x} = \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v} \Rightarrow \frac{dy}{dx} = uv' + u'v = -x^2e^{-x} + 2xe^{-x}$

$= e^{-x}\left[-x^2 + 2x\right]$

$= xe^{-x}\left[2 - x\right]$

(G) $y = \frac{e^x + e^{-x}}{x} = \frac{u(x)}{v(x)}$

$\overset{\text{quotient rule}}{\Rightarrow} \frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} = \frac{x\left[e^x - e^{-x}\right] - \left[e^x + e^{-x}\right](1)}{x^2}$

§4.5: Derivatives of Logarithmic Functions

What is the derivative of a logarithmic function? $f(x) = \log_a(x)$

i.e. $a^{f(x)} = x$

Since both sides of the equation are equal, lets take the derivative

$$\Rightarrow \frac{d}{dx}\left[a^{f(x)}\right] = \frac{d}{dx}[x] \Rightarrow (\ln a) f'(x)\, a^{f(x)} = 1$$

$$(\because a^{f(x)} = x) \Rightarrow (\ln a) f'(x)\, x = 1 \Rightarrow f'(x) = \frac{1}{x \ln a}$$

Derivative of $\log_a x$

$$\frac{d}{dx}[\log_a x] = \frac{1}{\ln a \cdot x} \qquad (a>0,\ a\neq 1)$$

Now when $a = e \Rightarrow \ln a = \ln e = 1$ so...

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

e.g. Find the derivative:

(A) $y = \ln 8x \Rightarrow \frac{dy}{dx} = \frac{1}{8x} \cdot \frac{d}{dx}[8x] = \frac{1}{8x} \cdot 8 = \boxed{\frac{1}{x}}$

(B) $\psi(x) = \ln(1+x^3) \Rightarrow \psi'(x) = \frac{1}{(1+x^3)} \cdot \frac{d}{dx}[1+x^3] = \boxed{\left(\frac{1}{1+x^3}\right) 3x^2}$

(C) $\xi(t) = \ln\sqrt{t^2+5}$

$$\Rightarrow \xi'(t) = \frac{1}{\sqrt{t^2+5}} \cdot \frac{d}{dt}\left(\sqrt{t^2+5}\right) = \frac{1}{\sqrt{t^2+5}} \cdot \frac{1}{2}(t^2+5)^{-1/2} \cdot 2t = \boxed{\frac{t}{t^2+5}}$$

Here's a formula if the chain rule still freaks you out...

Logarithm Derivative Rules

$$D_x[\log_a|x|] = \frac{1}{\ln a \cdot x} \quad ; \quad D_x[\log_a|g(x)|] = \frac{g'(x)}{(\ln a)\, g(x)}$$

$$D_x[\ln x] = \frac{1}{x} \quad ; \quad D_x[\ln|g(x)|] = \frac{g'(x)}{g(x)}$$

e.g. Find the derivative:

(A) $y = \ln(5x^3 - 2x)^{3/2}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(5x^3-2x)^{3/2}} \cdot \frac{3}{2}(5x^3-2x)^{1/2}(15x^2-2)$$

$$= \left[\frac{3}{2}(15x^2-2)\right]\frac{(5x^3-2x)^{1/2}}{(5x^3-2x)^{3/2}} = \frac{3}{2}\left[\frac{(15x^2-2)}{(5x^3-2x)^1}\right]$$

(B) $y = \underset{u}{e^{x^2}} \underset{v}{\ln x}$

$\xrightarrow{\text{product rule}}$ $y'(x) = uv' + u'v = \left(e^{x^2}\cdot\frac{1}{x}\right) + \left(2xe^{x^2}\right)\ln x$

(C) $y = \frac{e^x}{\ln x} = \frac{u}{v} \xrightarrow{\text{quotient rule}}$ $y' = \frac{vu' - uv'}{v^2} = \frac{(\ln x)e^x - e^x\left(\frac{1}{x}\right)}{(\ln x)^2}$ (simplify $\cdot\frac{x}{x}$)

$$= \frac{xe^x\ln x - e^x}{x(\ln x)^2}$$

(D) $\alpha(t) = \left(e^{2t} + \ln t\right)^3$

$\Rightarrow \alpha'(t) = 3\left(e^{2t} + \ln t\right)^2 \left(2e^{2t} + \frac{1}{t}\right)$

(E) $\zeta(x) = \sqrt{e^{-x} + \ln 2x} = \left(e^{-x} + \ln 2x\right)^{1/2}$

$2 \cdot \frac{1}{2x} = \frac{1}{x}$

$\Rightarrow \zeta'(x) = \frac{1}{2}\left(e^{-x} + \ln 2x\right)^{-1/2} \left[-e^{-x} + \frac{1}{x}\right]$